ON THE ABSTRACTION OF DRINKING-WATER FROM A CIRCULAR ISLAND IN SEA

by

D. H. Keuning*

1. *Introduction.*

In this paper a circular island in a sea of salt water is considered. The soil is composed of three layers, a top layer of sand with dunes emerging out of the sea, below this one a horizontal and infinitely thin, semi pervious layer, resting on a half space of sand. The top layer and the half space are well pervious to water, homogeneous and isotropic.

A part of the rainfall on the island flows off sideways into the sea, another part of it filters through the semi pervious layer, under which a bubble of fresh water is maintained, enclosed by salt water. The bubble is situated partly under the dunes and partly under the sea. The fresh water floats on the salt water like an iceberg does in the ocean, when flow effects are neglected.

Fresh water can be withdrawn above or below the semi pervious layer. In the first case the surface of the ground water (phreatic surface) will fall rapidly and substantially, by which the vegetation is threatened [1]. In the second case, when water is abstracted below the semi pervious layer, the phreatic surface will fall less and more gradually. But now the boundary between the fresh and salt water (boundary of salt) comes nearer to the island, by which it is possible that salt water is abstracted [1].

The last mentioned way of abstracting fresh water will be discussed in this paper. The steady flow is investigated.

2. *Formulation of the problem.*

A cylindrical coordinate system (y, r, φ) is chosen (fig. 2.1). The

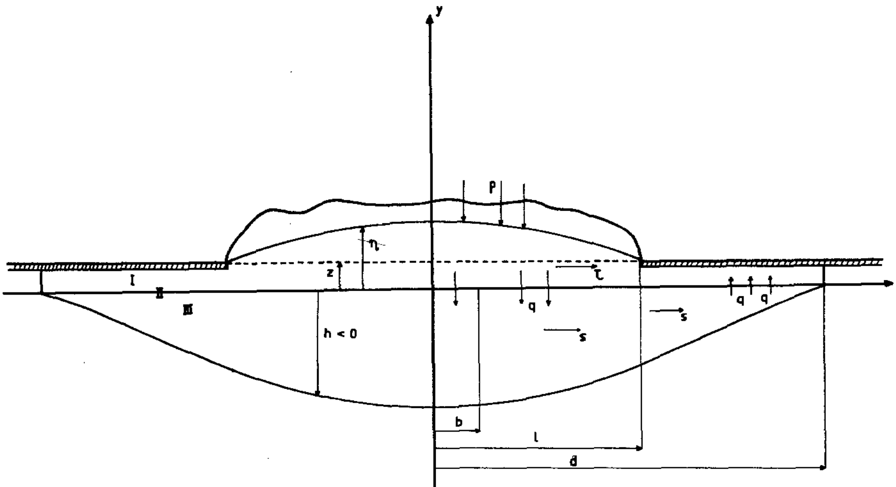

Fig. 2.1. Cross section of the island

* Mathematisch Instituut, Rijksuniversiteit Groningen, the Netherlands

(r, φ) plane coincides with the semi pervious layer (II), the y-axis is the axis of rotational symmetry of our problem, which is independent of φ.

The bubble of fresh water is situated below layer II in the half space (III) of well pervious sand. The radius of the bubble is denoted by d, the depth by $h = h(r) < 0$. Above the semi pervious layer is a layer of sand (I), with fresh water (density 1) for $r<d$ and with salt water for $r>d$, having a density $1 + \gamma$, where $\gamma = 0.024$. In the dunes the fresh water is bounded by the phreatic surface, the height of it is denoted by $\eta = \eta(r)$. The sea stretches on layer I for $r>1$, where 1 represents the radius of the island, and is assumed to be a shallow one.

The fresh water will be abstracted at $r = b$ below the semi pervious layer. The total amount of water which is withdrawn is $2\pi bQ$ units of volume per unit of time. The interval of values of r with $0<r<b$ is called region A, the interval $b<r<1$ region B and region C is the interval $1<r<d$.

The fresh water which filters through layer II downwards is partly abstacted and partly pressed upwards out of the bubble in region C. The latter part is carried off by the shallow sea together with the fresh water, which flows off sideways above the semi pervious layer. The salt water which surrounds the fresh water, is assumed to be at rest. No account is taken of low tide and high tide, for the sea-level we take an average value $y = z$.

Since the slopes of the phreatic surface and the boundary of salt will be small, the velocity of the fresh water is supposed to be horizontal and independent of y in the seperate layers I (for $r<1$) and III.

The potential p at a certain point is defined as the elevation to which fresh water would rise in an open tube sunk to the point in question, the elevation being measured from the sea-level.

The filtration velocity v equals the amount of ground water passing a unit of area perpendicular to the direction of flow per unit of time in the direction of flow ([2], page 11). The actual mean velocity of the water in the pores of the soil is much higher and depends on pore volume and on the structure of the soil. The filtration velocity in layer I for $r<1$ and in layer III satisfies Darcy's law ([2], page 13; [4], page 14), which becomes here

$$v = - k\frac{dp}{dr}, \tag{2.1}$$

where the permeability coefficient k characterizes the soil.

Since the velocity is assumed to be horizontal, the potential in layer I for $r<1$ and in layer III within the bubble of fresh water is independent of y. Hence the potential equals $\eta - z$ in the first region and $\gamma(z-h)$ in layer III. The potential is assumed to vanish in layer I for $1<r<d$, so that in this region the pressure difference needed for the vertical movement of the fresh water, and the difference of the densities of the fresh and the salt water in the shallow sea are neglected.

A discontinuity in the height of the phreatic surface, η, and the depth of the boundary of salt, h, must be preculed in connection with Darcy's law (2.1). Hence h and η are continuous functions of r, in particular $\eta(1) = z$.

Darcy's law gives for the flow in the regions A and B of layer I

$$\tau = -k\eta\frac{d\eta}{dr}, \tag{2.2}$$

where $\tau = \eta v$.

In the regions A and B of layer III Darcy's law yields

$$s = - \gamma kh\frac{dh}{dr}, \tag{2.3}$$

where $-\gamma \frac{dh}{dr}$ is the gradient of the potential and s equals - hv, h being negative.

The filtration velocity through layer II is proportional to the difference of the potentials directly above and below this layer, hence in the regions A and B holds

$$cq = \eta + \gamma h - (\gamma+1)z. \quad (2.4)$$

Here c is a constant called the resistance coefficient, $q = q(r)$ is the filtration velocity through layer II, while $\eta - z$ and $-\gamma(h-z)$ are the potentials above respectively below the semi pervious layer.

The law of continuity yields in the regions A and B of layer I

$$\frac{d\tau}{dr} + \frac{\tau}{r} = \rho - q, . \quad (2.5)$$

and in layer III

$$\frac{ds}{dr} + \frac{s}{r} = q . \quad (2.6)$$

Here ρ is the precipitation per unit of area and per unit of time reduced by the part which evaporates or is consumed by the vegetation.

Addition and integration with respect to r of (2.5) and (2.6) give the law of continuity applied to the layers I and II together. In layer I the equation reads

$$\tau + s = \tfrac{1}{2}\rho r, \quad (2.7)$$

in region B

$$\tau + s = \tfrac{1}{2}\rho r - \frac{bQ}{r}. \quad (2.8)$$

In region A the precipitation on the surface inside the circle with radius $r(r \leq b)$ equals the amount of water passing the generator of a cylinder with radius r between $y = \eta$ and $y = h$ (2.7). In region B the precipitation is reduced by the part which is abstracted (2.8).

In region C, where η equals z, (2.3) and (2.6) remain unaltered. Equation (2.4) changes into

$$cq = \gamma (h-z). \quad (2.9)$$

Equation (2.2) becomes an identity and (2.5) does not hold in region C, because an unknown amount of ground water is carried off by the shallow sea. Hence in region C (2.3), (2.6) and (2.9) are valid.

3. *Discussion of the equations, boundary and initial conditions.*

A differential equation for $h = h(r)$ will be derived in each of the three regions A, B and C seperately by eliminating the other dependent variables. First we determine the equation in region C. Elimination of q and s from (2.3), (2.6) and (2.9) gives a non linear differential equation of second order,

$$h \frac{d^2h}{dr^2} + \left(\frac{dh}{dr}\right)^2 + \frac{h}{r}\frac{dh}{dr} + \frac{h-z}{ck} = 0, \ 1 < r < d. \quad (3.1)$$

In order to find an equation in region B we eliminate τ and s from (2.2), (2.3) and (2.8), which yields

$$-\tfrac{1}{2}k\frac{d\eta^2}{dr} - \tfrac{1}{2}k\gamma\frac{dh^2}{dr} = \tfrac{1}{2}\rho r - \frac{bQ}{r}. \tag{3.2}$$

When we integrate with respect to r, (3.2) changes into

$$k\eta^2 + k\gamma h^2 + \tfrac{1}{2}\rho r^2 = 2bQ \ln\frac{r}{l} + D, \tag{3.3}$$

where D is an arbitrary constant. Another equation for η and h is derived by eliminating q and s from (2.3), (2.4) and (2.6),

$$\eta = -k\gamma c\left\{h\frac{d^2h}{dr^2} + \left(\frac{dh}{dr}\right)^2 + \frac{h}{r}\frac{dh}{dr}\right\} - \gamma h + (\gamma+1)z. \tag{3.4}$$

Elimination of η from (3.3) and (3.4) yields

$$h\frac{d^2h}{dr^2} + \left(\frac{dh}{dr}\right)^2 + \frac{h}{r}\frac{dh}{dr} + \frac{1}{k\gamma c}\Big\{\gamma h - (\gamma+1)z + \frac{1}{\sqrt{k}}\sqrt{D + 2bQ\ln\frac{r}{l} - \tfrac{1}{2}\rho r^2 - k\gamma h^2}\Big\} = 0, \quad b<r<l. \tag{3.5}$$

A same procedure is applied in order to determine the equation for h = h(r) in region A. In this region (3.3) has the form

$$k\eta^2 + k\gamma h^2 + \tfrac{1}{2}\rho r^2 = E, \tag{3.6}$$

where E is a constant of integration. Equation (3.4) is also valid in A, hence from (3.4) and (3.6),

$$h\frac{d^2h}{dr^2} + \left(\frac{dh}{dr}\right)^2 + \frac{h}{r}\frac{dh}{dr} + \frac{1}{k\gamma c}\Big\{\gamma h - (\gamma+1)z + \frac{1}{\sqrt{k}}\sqrt{E - \tfrac{1}{2}\rho r^2 - k\gamma h^2}\Big\} = 0, \quad 0<r<b. \tag{3.7}$$

In order to integrate (3.1), (3.5) and (3.7) we derive some initial and boundary conditions at the points r = d, r = l, r = b and r = 0. Since the equations are of second order, two conditions are required for each of the three regions. The conditions are given below and follow directly from considerations of continuity.

At the, still unknown, edge of the bubble, r = d, the conditions are

$$h = 0, \quad h\frac{dh}{dr} = 0. \tag{3.8}$$

At r = l they are

$$\left.\begin{aligned} h(l+0) &= h(l-0), \\ \frac{dh}{dr}(l+0) &= \frac{dh}{dr}(l-0), \end{aligned}\right\} \tag{3.9}$$

and at $r = b$

$$\left.\begin{aligned} h(b+0) &= h(b-0), \\ \frac{dh}{dr}(b+0) &= \frac{dh}{dr}(b-0) + \frac{Q}{k\gamma h(b)}. \end{aligned}\right\} \qquad (3.10)$$

From (3.10) we see that the slope of the bubble of fresh water has a discontinuity at $r = b$, caused by the abstraction of fresh water.

Since d is unknown, a seventh condition must be obtained in order to determine the solution. This one is found by applying the law of continuity to a small region in the neighbourhood of $r = 0$,

$$\frac{dh}{dr}(0) = 0. \qquad (3.11)$$

Finally we evaluate the constants of integration, D and E.
Since $\eta(1)$ equals z, (3.3) yields

$$D = kz^2 + k\gamma h^2(1) + \tfrac{1}{2}\rho l^2. \qquad (3.12)$$

The constant E is evaluated by substracting (3.3) from (3.6) at $r = b$ and by substituting (3.12),

$$E = kz^2 + k\gamma h^2(1) + \tfrac{1}{2}\rho l^2 + 2bQ \ln \frac{b}{l}. \qquad (3.13)$$

4. *The abstraction of water in absence of a semi pervious layer.*

When a semi pervious layer does not exist, the distinction between the layers I and III disappears. In this section the height of the phreatic surface, η, and the depth of the boundary of salt, h, are measured from the sea-level. We define the variable $s(r)$ as $(\eta-h)v$, v is the filtration velocity.

In the case with a semi pervious layer the bubble of fresh water is partly situated below the sea. In absence of this layer it is impossible that fresh water is present in the region $r>l$, since a discontinuity in the hydrostatic pressure must be precluded. The abstraction of fresh water takes place in the same manner as has been described in section 2. Again η and h have to be continuous functions of r, particularly $h(l) = \eta(l) = 0$, and the potential is independent of y within the bubble of fresh water.

For $y = \eta$ the potential equals η and for $y = h$ it is $-\gamma h$, hence for $0<r<l$ we have

$$\eta = -\gamma h. \qquad (4.1)$$

Darcy's law gives for $0<r<l$

$$s = -k(\eta-h)\frac{d\eta}{dr}. \qquad (4.2)$$

The law of continuity, applied to a cylinder with radius r, yields in region A,

$$s = \tfrac{1}{2}\rho r, \qquad (4.3)$$

and in region B

$$s = \tfrac{1}{2}\rho r - \frac{bQ}{r}. \qquad (4.4)$$

By eliminating η and s from (4.1), (4.2) and (4.3) we obtain a differential equation for h(r) in region A. In region B we obtain a similar equation by eliminating η and s from (4.1), (4.2) and (4.4). These equations can be integrated easily and the constants of integration can be determined by choosing r = 1 and r = b.

The solution in region A becomes

$$h = -\sqrt{\frac{\frac{1}{2}\rho(l^2 - r^2) + 2bQ \ln \frac{b}{l}}{\gamma k(\gamma+1)}}. \qquad (4.5)$$

and in region B

$$h = -\sqrt{\frac{\frac{1}{2}\rho(l^2-r^2) + 2bQ \ln \frac{r}{l}}{\gamma k(\gamma+1)}}. \qquad (4.6)$$

5. *The numerical integration and the results.*

The three differential equations (3.1), (3.5) and (3.7), which describe the function h(r) for the case with a semi pervious layer have to be integrated numerically. On that account we introduce the following dimensionless quantities,

$$\left.\begin{aligned} &\bar{h} = \frac{h}{l}, \quad \bar{r} = \frac{r}{l}, \quad \bar{b} = \frac{b}{l}, \quad \bar{d} = \frac{d}{l}, \\ &\bar{z} = \frac{z}{l}, \quad \bar{c} = \frac{kc}{l}, \quad \bar{\rho} = \frac{\rho}{k}, \\ &\bar{Q} = \frac{2bQ}{\rho l^2}, \quad \bar{D} = \frac{D}{kl^2}, \quad \bar{E} = \frac{E}{kl^2}. \end{aligned}\right\} \qquad (5.1)$$

Substitution of (5.1) into (3.1), (3.5) and (3.7) yields, after dropping the bars,

$$h\frac{d^2h}{dr^2} + \left(\frac{dh}{dr}\right)^2 + \frac{h}{r}\frac{dh}{dr} + \frac{h-z}{c} = 0, \quad 1<r<d, \qquad (5.2)$$

$$h\frac{d^2h}{dr^2} + \left(\frac{dh}{dr}\right)^2 + \frac{h}{r}\frac{dh}{dr} + \frac{1}{\gamma c}\Big\{\gamma h - (\gamma+1)z + \sqrt{D + Q\rho \ln r - \tfrac{1}{2}\rho r^2 - \gamma h^2}\Big\} = 0, \quad b<r<1, \qquad (5.3)$$

$$h\frac{d^2h}{dr^2} + \left(\frac{dh}{dr}\right)^2 + \frac{h}{r}\frac{dh}{dr} + \frac{1}{\gamma c}\Big\{\gamma h - (\gamma+1)z + \sqrt{E - \tfrac{1}{2}\rho r^2 - \gamma h^2}\Big\} = 0, \quad 0<r<b. \qquad (5.4)$$

The phreatic surface is a simple function of h and r according to (3.3) and (3.6), which read in the nondimensional form

$$\eta^2 + \gamma h^2 + \tfrac{1}{2}\rho r^2 = \rho Q \ln r + D, \quad b<r<1, \qquad (5.5)$$

$$\eta^2 + \gamma h^2 + \tfrac{1}{2}\rho r^2 = E, \quad 0<r<b. \tag{5.6}$$

When h is known the phreatic surface can be evaluated easily.

We have calculated the depth of the boundary of salt and the height of the phreatic surface for three values of Q, which is the abstracted part of the rainfall,

$$Q = 0, \ \frac{2}{17\pi} \approx 0.0375 \text{ and } \frac{6}{17\pi} \approx 0.1125, \tag{5.7}$$

and for three values of c,

$$c = 0.375, \ 1.25 \text{ and } 12.5. \tag{5.8}$$

For the remaining quantities we take

$$\left.\begin{aligned} \gamma &= 0.024, \\ b &= 0.1, \\ z &= 0.0075, \\ \rho &= 0.00017, \end{aligned}\right\} \tag{5.9}$$

which are, with $c = 0.375$, to a certain extent in agreement with the geological configuration of one of the Frisian islands in the Netherlands.

In order to start the computations the singularity, which equation (5.2) possesses at $r = d$, because there h vanishes, is eliminated by the substitution $h^2 = -y$. Then (5.2) changes into

$$\frac{d^2y}{dr^2} + \frac{1}{r}\frac{dy}{dr} + \frac{2(z+\sqrt{-y})}{c} = 0. \tag{5.10}$$

The pertaining initial conditions become

$$y = 0, \ \frac{dy}{dr} = 0, \ r = d. \tag{5.11}$$

In principle the numerical integration can be carried out as follows. For each assumed value of the radius $d(>1)$ of the bubble of fresh water we can integrate (5.10) from $r = d$ to $r = 1$. At the latter point the end values of h and $\frac{dh}{dr}$, which we obtain from the solution of (5.10), are the initial values for the solution of (5.3), on account of the condition of continuity (3.9). The constant D in (5.3) can be evaluated from the value h(1), making use of (3.12), which reads in the nondimensional form

$$D = z^2 + \gamma h^2(1) + \tfrac{1}{2}\rho \,. \tag{5.12}$$

Then we can integrate (5.3) from $r = 1$ to $r = b$.

At $r = b$ the initial values of h and $\frac{dh}{dr}$ for the solution of the equation (5.4) follow from the end values of h and $\frac{dh}{dr}$ of the solution of (5.3) according to the nondimensional form of condition (3.10),

$$\begin{aligned} h(b+0) &= h(b-0)\,, \\ \frac{dh}{dr}(b+0) &= \frac{dh}{dr}(b-0) + \frac{Q\,\rho}{2\gamma b h(b)}\,. \end{aligned} \tag{5.13}$$

The constant E can be evaluated from (3.13), which becomes in dimensionless form

$$E = z^2 + \gamma h^2(1) + \tfrac{1}{2}\rho + Q\rho \ln b. \quad (5.14)$$

Now we can integrate (5.4) from $r = b$ to the centre of the island.

It is the intention to carry out the integration to $r = 0$ and to determine such a value for d that condition (3.11) is satisfied. However equation (5.4) has a singularity at $r = 0$. In order to avoid this complication we integrate (5.4) from $r = b$ to $r = r_1 = 10^{-4}$. Now the radius of the bubble must be chosen in such a way that

$$\frac{dh}{dr}(r_1) = 0. \quad (5.15)$$

As method for the numerical integration we used Runge - Kutta's method. The accuracy of it is difficult to discuss. Therefore we compared some results of this method with the results obtained by the accurate method of Nordsieck [3], which needed however ten times as much computer time.

First we determined the behaviour of the solutions of the differential equations for some assumed values of the radius d of the bubble of fresh water. It appeared that for values which were larger than the actual radius, the function h(r) became strongly negative for decreasing values of r. By this the argument of the square root in (5.3), containing the term $-\gamma h^2$, became negative. For values of d which were smaller than the actual one, the boundary of salt tended to zero in the region A or B. In both cases we could not continue the integration. It turned out that there was an extremely small interval, in which we had to choose d in order to be able to reach the point $r = r_1$ with our integration procedure. The length of this interval varied from $5 \cdot 10^{-3}$ for $c = 12.5$ and $Q = 0$ to about 10^{-10} for $c = 0.375$ and $Q = \frac{6}{17\pi}$.

For the cases with $c = 0.375$ the solutions were so instable with respect to variations of d that we were not able to satisfy condition (5.15). It turned out that we could not obtain accurate values for the boundary of salt in the region $r_1 < r < 0.5$. The equations for these cases had also to be integrated in the direction of increasing values of r, starting with an assumed value for $h(r_1)$ and condition (5.15). It appeared that the solutions presented the same unstable behavior with respect to variations of $h(r_1)$ as the solutions, obtained from the integration in the direction of decreasing r, did with respect to variations of d. Only accurate values for h could be evaluated in the region $r_1 < r < 0.75$.

We obtained accurate values for h(r) from both directions of integrations in the interval $0.5 < r < 0.75$. It appeared that the values agreed up to six significant digits in this interval, hence the boundary of salt was obtained by pasting the solutions together.

The actual value of d and the actual value of $h(r_1)$ for the case that we integrated in reversed direction, where evaluated by a method of iteration, which could be developed, when we used the mentioned behaviour of the solutions of the differential equations.

We found that even in the most unstable cases the actual radius of the bubble of fresh water and the actual depth of the boundary of salt, which we evaluated with Runge - Kutta's method and Nordsieck's method agreed up to at least four significants digits. From this we conclude that Runge - Kutta's method is a reliable one for this problem. The reason for this unexpected reliability follows perhaps from the fact that the real boundary of salt is very smooth. When the initial value of d was different from the exact one, the results of both methods did not agreed at all.

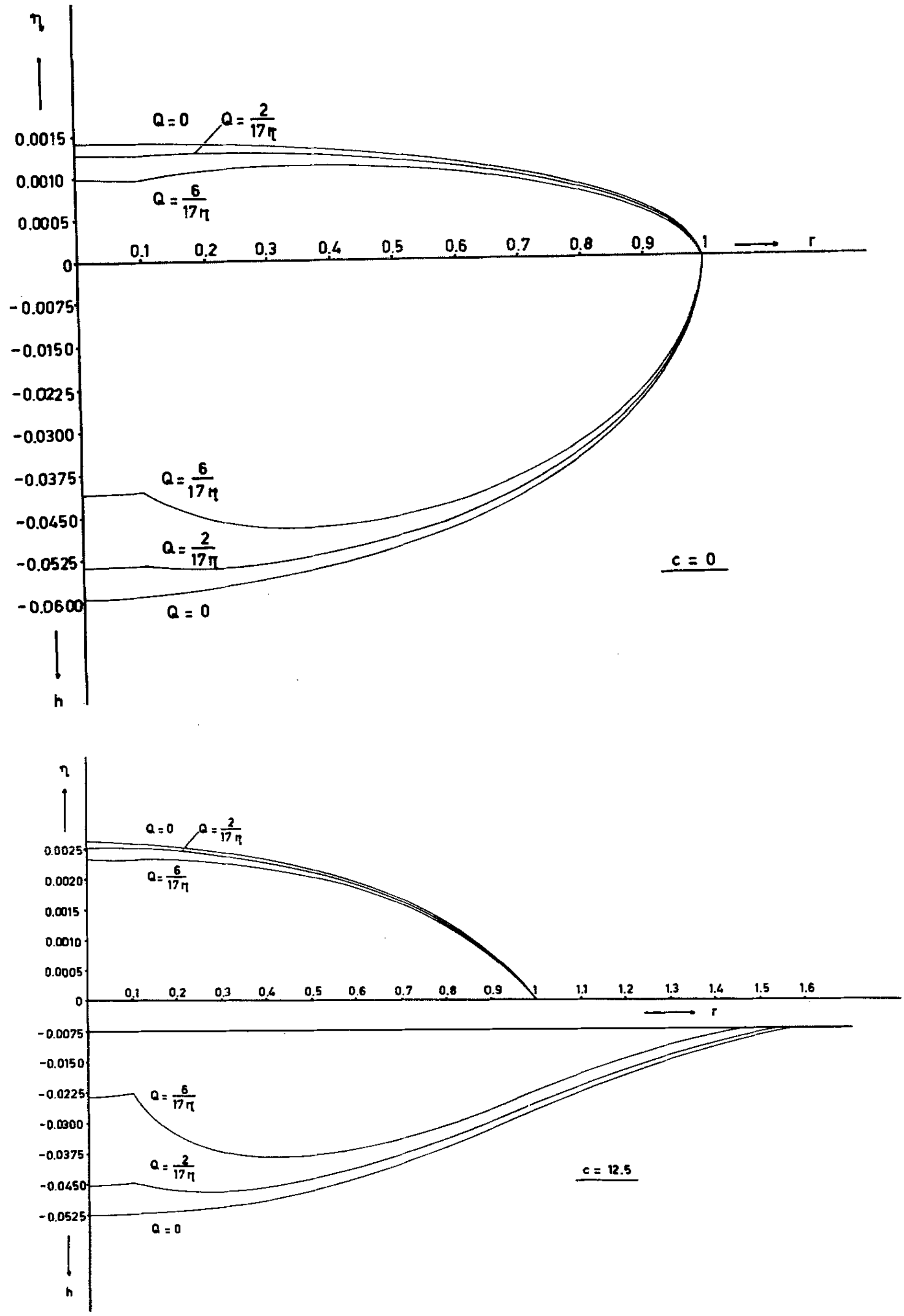

Fig. 5.1. The phreatic surfaces and the boundaries of salt for c = 12.5 and c = 0.

Besides for the values of c mentioned in (5.8) h and η were evaluated for the case of non existance of the semi pervious layers, denoted by $\bar{c}$ = 0. We remark that the case c tends to zero is not identical to this one. The phreatic surface and the boundary of salt are known explicitely in analytic form from section 4.

It turned out that the bubbles belonging to c = 0.375 and c = 1.25 differed only slightly from the bubble of fresh water with c = 0. Hence only the results for c = 12.5 and c = 0 are plotted in fig. 5.1.

ACKNOWLEDGMENT

The author is indebted to Prof. Dr. J. A. Sparenberg for helpful discussions during the course of this investigation.

REFERENCES

1. Mazure, J.P. Enkele vergelijkende berekeningen betreffende de gevolgen van boven en diep water onttrekking in het duingebied. Water, 14 - daags tijdschrift voor drinkwater - voorziening, Moorman's periodiekenpers, Den Haag, 27ste jaargang no.13, juni 1943.

2. Steady flow of ground water towards wells, Proceedings and informations no 10 of the Commitee for hydrological research TNO, The Hague, 1964.

3. Nordsieck, A., On numerical integration of ordinary differential equations, Mathematics of computation, vol.16, 1962. pg. 22 - 49.

4. Polubarinova - Kochina, P.Ya., Theory of ground water movement, Princeton University Press, Princeton, New Jersey, 1962.

[Received september 26, 1966]